Given Vectors Compute Magnitude, Direction, and Resultant Vectors

A vector can be thought of as a directed line segment. The vector \( \mathbf{u} = (2, 1) \) is shown below.

We often think of the starting point, A, as the origin, but as a directed line segment \( \mathbf{u} = (2, 1) \) can start at any point as shown in the second graph.

The direction of \( \mathbf{u} \) is indicated by going right 2 and up 1 or a slope, \( m = \tan \theta = \frac{1}{2} \). The direction is then the angle \( \theta = \tan^{-1} \left( \frac{1}{2} \right) \approx 26.57^\circ \).

The magnitude \( |\mathbf{u}| \) or length of \( \mathbf{u} \) can be computed using the Pythagorean Theorem.
\[
|\mathbf{u}| = \sqrt{a^2 + b^2} = \sqrt{2^2 + 1^2} = \sqrt{5}.
\]

Similarly for the vector \( \mathbf{v} = (1, 4) \) it has slope=4 so the direction is
\[
\theta = \tan^{-1}(4) \approx 75.96^\circ
\]

and magnitude, \( |\mathbf{v}| = \sqrt{1^2 + 4^2} = \sqrt{17} \).
We can add the vectors by adding components so we have
\[ w = u + v = \langle 2,1 \rangle + \langle 1,4 \rangle = \langle 3,5 \rangle , \]
w is called the resultant vector.

You might observe that w is the diagonal of a parallelogram determine by the sides \( u, v, u = w - v \), and \( v = w - u \). So we have \( u + v = w - v + v = w \).

Another operation you can perform with vectors is called scalar multiplication. So for our vector, \( u = \langle 1,2 \rangle \), \( au = a\langle 1,2 \rangle = \langle a,2a \rangle \) for any real number a.
**Objective:** Solve Problems Involving Vectors

Solve applications of vectors in the \( \mathbf{i} \mathbf{j} \) form.

The graph below shows the unit vectors: \( \mathbf{i} = \langle 1, 0 \rangle \) and \( \mathbf{j} = \langle 0, 1 \rangle \)

![Graph showing unit vectors](image)

Using scalar multiplication you can rewrite \( \mathbf{u} = \langle 2, 1 \rangle = 2\mathbf{i} + \mathbf{j} \) as can start at any point as shown in the second graph.

Consider the following physical applications:

1) **Forces are vector quantities.** An object of mass 1.00 kg is launched straight up into the air, and the magnitude of the force due to gravity acting on the object is therefore 9.81 N (newtons). This force acts on the rocket in a direction that is downward, and if we define “down” as the negative \( \mathbf{y} \) direction, we can write this vector as \( \mathbf{F}_g = -9.81 \mathbf{j} \). At the time of launch, wind blows from west to east, and the magnitude of the force of the wind acting on the object is 1.00 N. If we define “east” to be in the positive \( \mathbf{x} \) direction, we can write this vector as \( \mathbf{F}_w = 1.00 \mathbf{i} \).

   a. **Draw the vectors for the forces acting on the object.** Draw the resultant vector that represents the net force acting on the object. (Remember that the length of the arrow is representative of the magnitude of the vector.)

   ![Diagram showing forces](image)
Objective: Solve Problems Involving Vectors

b. Using the Pythagorean Theorem, find the magnitude of the resultant force acting on the object.

\[ |F| = \sqrt{F_g^2 + F_w^2} \]

\[ |F| = \sqrt{((-9.81)^2) + 1^2} \]

\[ |F| = 9.86 \text{ N} \]

c. Use trigonometry to find the direction of the resultant vector.

Because the two forces acting on the object are orthogonal, we have a simple right triangle with both the opposite and adjacent sides known. Given that the force of the wind has a direction of 0°, and the force of gravity has a direction of 270°, it is clear that our resultant force acts at an angle larger than 270°.

Utilizing the tangent function, we have:

\[ \tan \theta = \frac{-9.81}{1.00} \]

\[ \theta = \tan^{-1}(-9.81/1.00) \]

\[ \theta = 276° \]

(Note: We could also write the direction of this resultant force as “84° below the positive x-axis”.)

2) The electric field is a vector quantity. Two positive charges, charge 1 and charge 2, are separated by a certain distance. Charge 1 is at the origin, and charge 2 is at the (1,0). At the midpoint (0.50,0), the electric field due to charge 1 is \( E_1 = 1.00 \times 10^{-6} \text{ N/C } i \), where the unit N/C denotes “newton per coulomb”; force per unit charge. At the midpoint, the electric field due to charge 2 is \( E_2 = -2.00 \times 10^{-6} \text{ N/C } i \). What is the net electric field, \( \mathbf{E} \), at the midpoint?

Here, we have a one-dimensional problem, where both of the vectors lie on the x-axis. To find the resultant vector, \( \mathbf{E} \), we simply add the two components of the electric field due to each of the charges together, being sure to propagate the sign of each vector, which is indicative of the direction.

\[ \mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 \]

\[ \mathbf{E} = 1.00 \times 10^{-6} \text{ N/C } i - 2.00 \times 10^{-6} \text{ N/C } i \]

\[ \mathbf{E} = -1.00 \times 10^{-6} \text{ N/C } i \]
Objective: Solve Problems Involving Vectors

**Given Vectors Compute the Dot Product and Angle Between Two Vectors**

The **dot product** of \( \mathbf{u} \) and \( \mathbf{v} \) is simply the sum of the product of the component parts of each vector:  
\[
\mathbf{u} \cdot \mathbf{v} = 2 \cdot 1 + 1 \cdot 4 = 6
\]

We also have a convenient formula for finding the angle between two vectors:

\[
\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\| \mathbf{u} \| \| \mathbf{v} \|}
\]

![Diagram showing vectors and angle calculation](image)

So first we find the cosine of the angle between the two vectors and then use a calculator to compute it. Then use the inverse cos function on the calculator to compute the angle in degrees.

\[
\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\| \mathbf{u} \| \| \mathbf{v} \|} = \frac{6}{\sqrt{5} \cdot \sqrt{17}} = \frac{6}{\sqrt{85}}
\]

\[
\cos^{-1} \theta \approx 49.4^\circ
\]

Interestingly if our vectors are perpendicular the cosine is zero and the angle, 90.

In our graph we have \( \mathbf{3} \mathbf{i} = (3,0) \) and \( \mathbf{j} = (0,4) \). So the dot product is  
\[
\mathbf{u} \cdot \mathbf{v} = 3 \cdot 0 + 0 \cdot 4 = 0.
\]

**Work is a dot product of two vectors.** The dot product of two vectors is a scalar quantity, and work (which is given in units of energy) is the result of taking the dot product of the force vector,
Objective: Solve Problems Involving Vectors

$F$, and the displacement vector, $d$. In other words, $W = F \cdot d = |F||d|\cos \theta$. With a force of 25.0 N, a person pushes a box across the floor a distance of 2.00 m in the positive $x$ direction at an angle of 30.0° below the horizontal. **What is the work done on the box?**
Here, the word “horizontal” can be considered to be the $x$-axis. Therefore, the angle between the two vectors is 30.0°. In order to find the work done on the box, we take the dot product of the two vectors.

$$W = |F||d|\cos \theta$$
$$W = (25.0 \text{ N})(2.00 \text{ m})\cos 30$$
$$W = 43.3 \text{ Nm} = 43.3 \text{ J}.$$  

*(J (joules) is a unit of energy)*

**Magnetic flux is a dot product of two vectors.** The flow of magnetic flux through an area is given by taking the dot product of the magnetic field vector, $B$, and the vector of the surface area through which the field flows, $A$. Mathematically, this is given as $\Phi = B \cdot A = |B||A|\cos \theta$, where the angle is between the direction of the magnetic field and the normal to the surface of the area through which the magnetic flux flows. **If the magnetic flux through a square loop of area 0.0625 $m^2$ is equal to 0.0380, and the magnetic field has a magnitude of 0.650 $T$, what is the angle between the magnetic field and the normal to the surface?**

$$\cos \theta = \Phi/(BA)$$
$$\theta = \cos^{-1}(0.0380/(0.650\times0.0625))$$
$$\theta = 20.7^\circ$$