Derive and Identify the Law of Sines.

Consider the triangle with altitude h.

Writing expressions for \( \sin C \) and \( \sin B \) we have

\[
\sin C = \frac{h}{b} \quad \text{and} \quad \sin B = \frac{h}{c}.
\]

Solving for \( h \) for each equation above:

\[
h = b \sin C \quad \text{and} \quad h = c \sin B.
\]

Setting the expressions for \( h \) equal we have

\[
c \sin B = b \sin C \quad \text{or} \quad \frac{\sin B}{b} = \frac{\sin C}{c}
\]

This gives us the Law of Sines for all angles and all sides.

\[
\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}
\]

or

\[
\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}
\]
Derive and identify the Law of Cosines.

Consider the triangle with altitude $h$.

Using both right triangles and the Pythagorean Theorem to write two different expressions for $h^2$, we get

$$h^2 = c^2 - x^2 \quad \text{and} \quad h^2 = b^2 - (a - x)^2$$

Setting the expressions for $h^2$ equal we have

$$c^2 - x^2 = b^2 - (a - x)^2$$
$$b^2 = c^2 - x^2 + (a - x)^2$$
$$b^2 = c^2 - x^2 + a^2 - 2ax + x^2$$
$$b^2 = c^2 + a^2 - 2ax$$

Now using $\cos B$ to find an expression for $x$

$$\cos B = \frac{x}{c}$$
$$x = c \cos B$$

and substituting this into our equation above we have

$$b^2 = c^2 + a^2 - 2ac \cos B.$$

This is the Law of Cosines.

<table>
<thead>
<tr>
<th>$b^2 = a^2 + c^2 - 2ac \cos B$</th>
<th>You can also solve for the angle.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a^2 = b^2 + c^2 - 2bc \cos A$</td>
<td>$B = \cos^{-1}\left(\frac{a^2 + c^2 - b^2}{2ac}\right)$</td>
</tr>
<tr>
<td>$c^2 = a^2 + b^2 - 2ab \cos C$</td>
<td></td>
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